SOME PROCEDURES FOR DETERMINING SUBJECTIVE PROBABILITIES BY SEQUENTIAL CHOICES*

by

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1. Introduction.

Definitions of "subjective" or "personal" probability, such as that of Savage, involve hypothetical preferences or choices of a subject which carry information about subjective probability values. For example, Savage (1954, p. 28) says:

> We therefore address him thus: "We see you are about to open those eggs. If you will be so cooperative as to guess that one or the other egg is good, we will pay you a dollar, should your guess prove correct. If incorrect you and we are quits, except that we will in any event exchange your two eggs for two of guaranteed goodness." If under these circumstances the person stakes his chances on the brown egg, it seems to me to correspond well with ordinary usage to say that it is more probable to him that the brown one is good than that the white one is.

By replacing the breaking of one egg (outcomes "good" or "bad") with the toss of a fair coin (outcomes "heads" or "tails") we may determine whether the subjective probability of the remaining egg being good is $\geq 1/2$ or $\leq 1/2$. If further choices are then offered, the subjective probability presumably can be restricted to narrower limits. We find however that certain difficulties arise in the choice of stake at each stage of the procedure. In short, an apparently dishonest choice may in some cases be advantageous to the subject. The present paper is concerned with the characterization of procedures which "encourage honesty."

The problems have counterparts in previously studied methods of "payoff functions" for evaluating the performance of probability appraisers. Such payoff functions were first suggested by Good (1952), and have since been studied for example by McCarthy (1956), deFinetti (1962) and Winkler (1967). In these studies, the appraiser is offered a prize (or possibly is penalized) by a reward depending on his estimated or appraised value, say $\hat{p}$, and on the actual outcome of the event whose probability is appraised. Such payoffs are said to "encourage honesty" if the expected payoff calculated from any hypothetical true value of p is maximized when $\hat{p} = p$. McCarthy (1956) characterizes payoff functions having this property.

In the present paper sequential procedures are studied in which, for simplicity, each choice divides in half the range of possible values of the probability. These procedures are described in Section 2. Section 3 gives expressions for the accumulated expected payoff. In Section 4 we define payoffs which "encourage honesty" to be those having the property that the honest choice by the subject at each stage gives the largest accumulated expected payoff when the true probability is known. Sections 5 and 6 establish some properties of expected payoffs, including characterizations of payoffs which encourage honesty for finite sequential procedures. The infinite case is discussed briefly in Section 7.

2. Description of the procedure.

Let A denote an event having unknown probability p and let B_r denote an event with known probabilities r. (We assume such exist for $r = 1/2, 1/4, 3/4, 1/8, 3/8, 5/8$, etc. Of course the existence of a single fair coin which can be tossed repeatedly implies this.) The appraiser is offered a sequence of choices of prospects. At any step Prospect A is a payoff of $g(r)$ if A occurs and Prospect B is a payoff of $g(r)$ if B_r occurs. It remains to describe the sequence $\{r_n\}$ of r-values and to choose $g(r)$. This is to be done in such a way that the best choices for the appraiser when p is actually known will allow us to deduce the value of p from his choices.

Consider the following particular choice. At Step 1, $r = r_1 = 1/2$. At Step 2, $r = r_2 = 1/4$ or $3/4$ according as Prospect B or A was chosen at Step 1. Similarly at Step n, $r = r_n$ will be either $r_{n-1} - 2^{-n}$ or $r_{n-1} + 2^{-n}$ according as the preceding choice was B or A. Clearly the idea here is to obtain a sequence of r values converging to the true p when it is known, or to the appraised value $\hat{p}$ in any case. The appraiser will presumably attempt to maximize his expected payoff which he will calculate using $\hat{p}$ rather than p when the latter is unknown. Of course at Step 1 he then chooses A if $\hat{p} > 1/2$ provided he only considers the payoff determined by this first choice. If however the rules for the subsequent steps are known to him, the "dishonest" choice of B when $p > 1/2$ could actually increase his expectation at later stages. For this reason the choice of values $g(r)$ is relevant to an honest appraisal.

3. Expressions for expected payoff functions.

Let $a_j = -1$ if Prospect B is chosen at the jth step and $a_j = +1$ if A is chosen. Then r_n can be expressed by

$$r_n = \frac{1}{2} + \sum_{j=1}^{n-1} 2^{-j-1} a_j, \qquad n = 2, 3, \ldots, \tag{3.1}$$

and $r_1 = 1/2$.

There are slightly troublesome difficulties which arise when the appraiser is indifferent between the two prospects. For example, if $p = 1/2$ he is indifferent between A and B. However in the case when $n \to \infty$ we can still obtain $r_n \to 1/2$ whether $r_2 = 1/4$ or $3/4$. It would be desirable here to choose the values $g(r)$ so that the expected payoff does not depend on which choice is made in any case where the appraiser is indifferent. The extent to which this can be accomplished will be considered below.

It is convenient to define

$$(3.2) \qquad C_n = \{x \mid x = 2^{-n}k,\ k = 1, 3, \ldots, 2^n-1\}$$

$$(3.3) \qquad D_n = \bigcup_{j=1}^{n} C_j\ ;$$

$$(3.4) \qquad D = D_\infty .$$

Then our payoff function $g(r)$ is to be defined for all $r \in D$ (although we will also consider finite procedures requiring only $r \in D_n$).

At Step n the payoff is either $g(r_n)$ with probability r_n if B is chosen ($a_n = -1$) or $g(r_n)$ with probability p if A is chosen ($a_n = +1$). Thus the increment in the expected payoff is $r_n g(r_n)$ if $a_n = -1$ or $pg(r_n)$ if $a_n = +1$. The cumulative expected payoff at the <u>nth</u> step depends on p and on the partial sequence of choices

$$(3.5) \qquad G_n = (a_1, a_2, \ldots, a_n)$$

and can be expressed as

$$(3.6) \qquad H_n(G_n, p) = F_n(G_n) + pG_n(G_n)$$

where

(3.7) $$F_n(\mathcal{G}_n) = \sum_{\substack{1 \le k \le n \\ a_k = +1}} r_k g(r_k)$$

(3.8) $$G_n(\mathcal{G}_n) = \sum_{\substack{k \le k \le n \\ a_k = +1}} g(r_k).$$

Now let r be any value $0 < r < 1$ and $r \notin D_n$. Then r determines a unique binary expansion up to the <u>nth</u> term, and so we may replace (3.5) by

(3.9) $$H_n(r, p) = F_n(r) + pG_n(r)$$

understanding that r determines a unique $\mathcal{G}_n$. From (3.9) we see that the cumulated expected payoff $H_n(r, p)$ is linear in p and a step function in r whose discontinuities occur at the points of D_n, that is at multiples of 2^{-n}, or we may say at exactly those points of $(0, 1)$ where $H_n(\cdot, p)$ is not defined.

It is straightforward to calculate the jump in $H_n(\cdot, p)$ at each discontinuity. Consider first Case 1: $r = k/2^n$ where k is odd (that is $r \in C_n$). We find

(3.10) $$F_n(r-) - F_n(r+) = rg(r)$$

(3.11) $$G_n(r+) - G_n(r-) = g(r).$$

For Case 2, we express any other point r of discontinuity $(r \in D_{n-1} = D_n - C_n)$ uniquely in the form $r = 2^{-s}k$ where k is odd and $s < n$. For these we find

$$(3.12) \qquad F_n(r-) - F_n(r+) = rg(r) - \sum_{j=1}^{n-s} (r + 2^{-s-j})g(r + 2^{-s-j})$$

$$(3.13) \qquad G_n(r+) - G_n(r-) = g(r) - \sum_{j=1}^{n-s} g(r - 2^{-s-j}).$$

4. Payoffs which encourage honesty.

We assume that the appraiser knows the rules by which the payoffs are to be made, including the payoff values $g(r)$. The range of values of $H_n(r, p)$ (where $0 < p < 1$ is fixed and $r \varepsilon [0, 1] - D_n$) is the set of all possible expected payoffs for the appraiser at time n. From (3.6) through (3.9) the appraiser can calculate his expected payoff from the value of r which determines the sequence G_n and from a presumed value of p.

A payoff function would encourage honesty if

$$(4.1) \qquad H_n(p, p) > H_n(r, p) \qquad \text{for all } r \neq p.$$

But since $H_n(r, p)$ is a step function in r we cannot actually achieve inequality for all $r \neq p$. If $p \neq 2^{-n}k$ (k odd), let I_{kn} be the interval $(2^{-n}k, 2^{-n}(k+1))$ (k odd) which contains p. We require

$$(4.2) \qquad H_n(p, p) > H_n(r, p) \qquad \text{all } r \notin I_{kn},$$

(for all $p \neq 2^{-n}k$). Any payoff $g(r)$ such that (4.2) is satisfied will be said to "encourage honesty" at the nth stage. For p values on the boundary points $2^{-n}k$ no additional requirement is needed since (4.2) is strong enough to imply appropriate behavior. More specifically, it can be shown that when $p = 2^{-n}k$

(4.3) $H_n(p+, p) = H_n(p-, p) \geq H_n(r, p)$ all $r \in [0, 1] - D_n$.

The proof uses (4.2) and continuity of $H_n(r, p)$ as a function of p. Hence if (4.2) holds and $p \in D_n$ then the appraiser's maximum expected payoff is $H_n(p+, p) = H_n(p-, p)$. For simplicity, extend the domain of definition of $H_n(r, p)$, $F_n(r)$, and $G_n(r)$ to $r \in [0, 1]$ by right-continuity. Then (4.2) is equivalent to

(4.4) $H_n(p, p) \geq H_n(r, p)$

with equality only when both p and r belong to the same interval $(2^{-n}k, 2^{-n}(k+1))$ or when $p = 2^{-n}k$ and $r \in (2^{-n}(k-1), 2^{-n}(k+1))$.

5. Properties of expected payoffs.

We first consider properties of H_n defined by (3.9) which hold for rather arbitrary F_n and G_n.

Theorem 1. Let $F(\cdot)$ and $G(\cdot)$ be defined on $(0, 1)$; let

(5.1) $H(r, p) = F(r) + pG(r)$

for $r, p \in (0, 1)$ and let

(5.2) $H(p, p) \geq H(r, p)$ all $r, p \in (0, 1)$.

Then (i) F is nonincreasing and G is nondecreasing. (ii) For fixed p, $H(r, p)$ is nonincreasing for $r > p$ and nondecreasing for $r < p$. (iii) $H(p, p)$ is continuous.

Proof. (i) Using (5.2) twice we have

$$\begin{aligned} F(p) + G(p)p &\geq F(r) + G(r)p = F(r) + G(r)r - G(r)(r-p) \\ &\geq F(p) + G(p)r - G(r)(r-p) = F(p) + G(p)p + G(p)(r-p) \\ &\qquad - G(r)(r-p). \end{aligned}$$

Thus $(G(p) - G(r))(r-p) \leq 0$, which implies G is nondecreasing. Assuming $p < r$, using (5.2) again, and $G(p) \leq G(r)$

$$\begin{aligned} F(p) + pG(p) &\geq F(r) + pG(r) \\ &\geq F(r) + pG(p) \end{aligned}$$

completing the proof of (i). (ii) If $p \geq r \geq s$, by (5.2) and (i),

$$\begin{aligned} F(r) + pG(r) &= F(r) + rG(r) + (p-r)G(r) \geq F(s) + rG(s) + (p-r)G(s) \\ &= F(s) + pG(s). \end{aligned}$$

If $p \leq s \leq r$

$$\begin{aligned} F(r) + pG(r) &= F(r) + sG(r) + (p-s)G(r) \leq F(s) + sG(s) + (p-s)G(s) \\ &= F(s) + pG(s) \end{aligned}$$

which proves (ii). (iii) Let $\{y_n\}$ be either an increasing or a decreasing sequence with $\lim y_n = p$. Then both $\lim F(y_n)$ and $\lim G(y_n)$ exist, and using (16) twice

$$\begin{aligned} F(p) + pH(p) &\geq \lim \{F(y_n) + pG(y_n)\} \\ &= \lim \{F(y_n) + y_n G(y_n)\} \\ &\geq \lim \{F(p) + y_n G(p)\} \\ &= F(p) + pG(p) \qquad \text{qed.} \end{aligned}$$

<u>Theorem 2</u>. Let $F(\cdot)$ and $G(\cdot)$ be defined on $(0, 1)$ and let H be defined by (5.1). Then (5.2) holds if and only if G is nondecreasing and

$$(5.3) \qquad F(r) = \int_r^1 z \, dG(z) + c$$

for some constant c.

<u>Proof</u>. Assume (5.3) and G nondecreasing. Then

$$H(r, r) - H(s, r) = \begin{cases} \int_r^s (z-r)dG(z) \geq 0 & \text{if } s > r \\ \int_s^r (r-z)dG(z) \geq 0 & \text{if } s < r \end{cases}$$

so that (5.2) holds. Now assume (5.2). Then G is nondecreasing by Theorem 1. Let $h(r)$ be defined by

$$(5.4) \qquad F(r) = h(r) + \int_r^1 zdG(z).$$

Then (5.2) implies

$$(5.5) \qquad F(r) - F(s) \geq r\{G(s) - G(r)\},$$

from which we get

$$(5.6) \qquad h(r) - h(s) \geq \int_s^r (z-r)dG(z)$$

where $\int_s^r$ means $-\int_r^s$ when $r < s$. For any $\varepsilon > 0$ we can find a decreasing or increasing sequence $\{x_k\}_{k=1}^n$ with $x_1 = s$, $x_n = r$ and

$$(5.7) \qquad |\int_s^r zdG(z) - \sum_{k=1}^{n-1} x_k \int_{x_k}^{x_{k+1}} dG(z)| < \varepsilon.$$

Then using (5.6) on each term

$$(5.8)\qquad h(r) - h(s) = \sum_{k=1}^{n-1} \{h(x_{k+1}) - h(x_k)\}$$

$$\geq \sum_{k=1}^{n-1} \int_{x_k}^{x_{k+1}} (z - x_k)dG(z)$$

$$\geq - \epsilon.$$

Hence $h(r) - h(s) \geq 0$. But since r and s are arbitrary we must have $h(s) - h(r) \geq 0$. Therefore $h(\cdot)$ is constant. qed.

<u>Corollary 1</u>. If F and G are right-continuous step functions, then necessary and sufficient conditions for (5.2) are: (i) G is non-decreasing; (ii) $H(p, p)$ is continuous for all p.

<u>Proof</u>. Let $\{r_k\}_{k=1}^{\infty}$ be the points where either F or G is discontinuous. Then a necessary and sufficient condition for (5.3) is

$$(5.9)\qquad F(r_k-) - F(r_k+) = r_k[G(r_k+) - G(r_k-)] \qquad \text{all } k,$$

and this is equivalent to (ii).

<u>Corollary 2</u>. If (5.2) holds and $H(p, p) = H(r, p)$ for some $r, p \in [0, 1]$ then $F(p) = F(r)$ and $G(p) = G(r)$.

<u>Proof</u>. If $H(p, p) = H(r, p)$ then

$$(5.10)\qquad F(p) - F(r) = p[G(r) - G(p)].$$

If (5.2) holds, then by Theorem 1, G is nondecreasing and by (5.10),

$$\int_p^r zdG(z) = \int_p^r pdG(z), \text{ or}$$

$$\int_p^r (z-p)dG(z) = 0.$$

But the above integral is positive unless $G(r) = G(p)$, in which case $F(r) = F(p)$.

6. Properties of payoffs which encourage honesty.

We now apply the preceding results to determine properties of the payoffs $g(r)$ which will encourage honesty.

Lemma 1. Let F_n, G_n, H_n be defined by (3.9) through (3.13). Then $H_n(p, p)$ is continuous if and only if

$$(6.1) \qquad \sum_{j=1}^{n-s} (r + 2^{-s-j})g(r + 2^{-s-j}) = r \sum_{j=1}^{n-s} g(r - 2^{-s-j}) \quad \text{for all} \quad r \in C_s$$

and for all $s = 1, 2,\ldots, n-1$.

Proof. For $r \in C_n$, $H_n(p, p)$ is continuous by (3.10), (3.11). For $r \in D_{n-1} = D_n - C_n$ continuity of $H_n(p, p)$ is equivalent to (5.9) (with subscripts n added) and (3.12) and (3.13) give (6.1).

Theorem 3. Let F_n, G_n, H_n be defined by (3.9) through (3.13) and the assumption of right-continuity. Then necessary and sufficient conditions for (4.2) are (6.1) and

$$(6.2) \qquad g(r) > \sum_{j=1}^{n-s} g(r - 2^{-s-j}) \quad \text{all} \quad r \in C_s,$$

for $s = 1, 2,\ldots, n-1$.

Proof. We can use the theory of Section 5 with subscript n added to F, G, and H. Using Corollary 2 it is easy to see that (4.2) is equivalent to (5.2) with the additional condition that $G_n(p+) \neq G_n(p-)$ for $p \in D_n$. Hence by Corollaries 1 and 2 we have (4.2) holds if and only if the following three conditions hold:

(i) G_n is nondecreasing

(ii) $H_n(p, p)$ is continuous

(iii) $G_n(p+) \neq G_n(p-)$ for $p \in D_n$.

By Lemma 1, condition (ii) is equivalent to (6.1). Since G_n is a step function with jumps given by (3.13), we have conditions (i) and (iii) equivalent to (6.2).

Theorem 4. Let F_n, G_n, H_n be defined by (3.9) through (3.13) for $n = 1, 2, \ldots$. Then necessary and sufficient conditions for (4.2) to hold for all positive integers n are that for every $k = 3, 5, \ldots, 2^m-1$,

$$(6.3) \qquad g(2^{-m}k) = \frac{k-1}{k} g(2^{-m}(k-2)) = h(k)g(2^{-m}) \qquad m = 2, 3, \ldots,$$

$$(6.4) \qquad g(2^{-m}) \geq \frac{1}{h(k)} \sum_{j=1}^{\infty} h(2^j k - 1) g(2^{-m-j}) \qquad m = 1, 2, \ldots,$$

where

$$(6.5) \qquad h(k) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}(k+1)\right)/\Gamma\left(\frac{1}{2}k+1\right) = \frac{2\cdot 4\cdots(k-3)(k-1)}{1\cdot 3\cdots(k-2)k}.$$

Proof. Suppose (6.3) and (6.4) hold. For given r, s, j, define k by $k = 2^m r + 1$ where $m = s + j$. Then $2^{-m}k = r + 2^{-s-j}$, $2^{-m}(k-2) = r - 2^{-s-j}$, and (6.3) implies termwise equality of the sums in (6.1). To prove (6.2), for given $r \in C_s$ define m,k by $m = s$ and $2^{-m}k = r$. Then by (6.3) $h(k)g(2^{-m}) = g(r)$ and the corresponding terms under the right hand summations in (6.2) and (6.4) are equal (by (6.3)). Inequality in (6.2) follows from dropping terms (all positive) beyond $j = n - s$. Thus (6.1) and (6.2) hold, and by Theorem 3, these imply (4.2). Now suppose (4.2) holds, or equivalently (6.1) and (6.2) hold. We will prove (6.3) by induction on m.

If $m = 2$ then (6.1) gives $(r = \frac{1}{2})$

$$(6.6) \qquad \frac{3}{4}\, g(\tfrac{3}{4}) + \frac{1}{2}\, g(\tfrac{1}{2}) = \frac{1}{2}[g(\tfrac{1}{4}) + g(\tfrac{1}{2})] \text{ or } g(\tfrac{3}{4}) = \frac{2}{3}\, g(\tfrac{1}{4})$$

which establishes (6.3) for $m = 2$. Now suppose (6.3) holds for $m = 2, 3, \ldots, m_0-1$. We will show it holds also for $m = m_0$. For any $k = 3, 5, \ldots, 2^{m_0}-1$, let $r = (k-1)2^{-m_0}$. Then $r \in C_s$ where $s < m_0$. By the induction hypothesis the summations of (6.1) are equal term by term for $s + j < n$, that is for all terms except the last. Hence by (6.1) the last terms are equal. That is, $(r+2^{-m})g(r+2^{-m}) = rg(r-2^{-m})$, or $2^{-m}kg(2^{-m}k) = 2^{-m}(k-1)g(2^{-m}(k-2))$. This proves (6.3). The proof of (6.4) involves substituting terms given by (6.3).

A simple sufficient condition can be given for the condition (6.4) of Theorem 4. For h defined by (6.5) we have

$$(6.7) \qquad \frac{h(2^j k-1)}{h(k)} = \frac{k+1}{k+2} \cdot \frac{k+3}{k+4} \cdots \frac{2^j k - 2}{2^j k - 1} < 1 .$$

Now suppose that

$$(6.8) \qquad g(2^{-m}) \geq \sum_{j=1}^{\infty} g(2^{-m-j}) \qquad m = 1, 2, \ldots .$$

Then (6.7) and (6.8) imply (6.4). Thus (6.3) and (6.8) are sufficient conditions for (4.2). A further simplification is the restriction

$$(6.9) \qquad g(2^{-m-1}) \leq \frac{1}{2}\, g(2^{-m}) \qquad m = 1, 2, \ldots$$

which implies (6.8). This leads us to a particular solution obtained by taking equality in (6.9) for each m. Arbitrarily taking $g(\frac{1}{2}) = \frac{1}{2}$ we get $g(\frac{1}{4}) = \frac{1}{4}$, $g(\frac{3}{4}) = \frac{2}{3} \cdot \frac{1}{4}$, $g(\frac{1}{8}) = \frac{1}{8}$, $g(\frac{3}{8}) = \frac{2}{3} \cdot \frac{1}{8}$, $g(\frac{5}{8}) = \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{1}{8}$, $g(\frac{7}{8}) = \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdot \frac{1}{8}$, $g(\frac{1}{16}) = \frac{1}{16}$, etc.

7. Payoffs which encourage honesty in the limit.

Consider the case of an infinite sequence of choices $(n \to \infty)$. Then the expected payoff for $r \notin D$ is $\lim_{n\to\infty} H_n(r, p)$. If (4.2) holds for each n then

$$(7.1) \qquad H(p, p) \geq H(r, p) \qquad \text{for all } p \notin D, r \notin D.$$

Thus the infinite procedure <u>almost</u> encourages honesty, where we must say "almost" because of the possibility of equality in some cases when $r \neq p$. However it can be shown that (6.3) plus strict inequality in (6.4) plus existence of $\lim H_n(r, p)$ implies strict inequality in (7.1) whenever $r \neq p$.

Presumably (7.1) could hold for infinite procedures which do not encourage honesty for each n. We hope to consider such cases in a later report.

8. References.

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